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W-geometry from chiral embeddings

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Recent work by Y. Matsuo and the present author is summarized. It is shown that, classically, the conformal Toda equations associated with the simple lie algebras $A_n \equiv \mathrm{sl}(n+1)$ follow from the embedding of particular 2D surfaces in CP^n. Since these Toda theories provide Noether realizations of W-symmetries, this gives the geometrical interpretation of the corresponding two-dimensional physics.

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1. Introduction

In many ways, W-algebras are natural generalizations of the Virasoro algebra. They were first introduced as consistent operator algebras involving operators of spins higher than two [3]. Moreover, the Virasoro algebra is intrinsically related with the Liouville theory which is the Toda theory associated with the Lie algebra A_1, and this relationship extends to W-algebras which are in correspondence with the family of conformal Toda systems associated with arbitrary simple Lie algebras [4]. Another point is that the deep connection between Virasoro algebra and KdV hierarchy has a natural extension [5] to W-algebras and higher KdV (KP) hierarchies [6,7].

On the other hand, W-symmetries exhibit strikingly novel features. First, they are basically non-linear algebras. Since the transformation laws of primary fields contain higher derivatives, products of primaries are not primaries at the classical level. Naive tensor products of commuting representations do not form representations. A related novel feature is that W-algebras generalize the diffeomorphisms of the circle by including derivatives of degree higher than one. Going beyond linear approximation (tangent space) is a highly non-trivial step.

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Taking higher order derivatives changes the shape of the world-sheet in the target space, thus W-geometry should be related to the extrinsic geometry of the embedding. Finally, Virasoro algebras are notoriously related to Riemann surfaces. The W-generalization of the latter notion is a fascinating problem.

From the viewpoint of two-dimensional (2D) physics, the Liouville theory arises from 2D gravity in the conformal gauge. Its Toda generalizations should be related similarly to what is called W-gravity, a theory which is far from being understood since its only known in special gauges. These notes describe the geometrical meaning of the Toda theories associated with $A_n \equiv \mathrm{sl}(n+1)$, following refs. [1,2], to which we refer for details. It will be shown that the Toda equations arise from the extrinsic geometry of chiral surfaces in n-dimensional complex projective spaces. This is a first step towards the geometrical understanding of W-gravity and W-symmetries. Moreover, the Toda equations are part of the Toda hierarchy, and we will obtain a geometrical interpretation of the latter as well.

2. Geometry of chiral surfaces

2.1. CHIRAL EMBEDDINGS

The basic objects are two-dimensional surfaces in Kähler manifolds. We shall only consider C^n here explicitly. The case of CP^n is treated by using homogeneous coordinates in C^{n+1}. We call z and $\overline{z}$ the surface parameters. One may think of z as an ordinary complex number, in which case the parametrization is Euclidean, and $\overline{z}$ is the complex conjugate of z; or take z and $\overline{z}$ to be real, in which case $x_0 \equiv (z+\overline{z})/2$, and $x_1 \equiv (z-\overline{z})/2$ are coordinates of the Minkowski type. The adjective chiral means function of a single variable z or $\overline{z}$ (if z is a complex variable this is equivalent to analytic or anti-analytic). We parametrize C^n by coordinates X^A, $\overline{X}^{\overline{A}}$, $1 \leq A, \overline{A} \leq n$. A *chiral embedding* is defined by equations of the form

$$\begin{aligned} X^A &= f^A(z), \quad A = 1, \ldots, n, \\ \overline{X}^{\overline{A}} &= \overline{f}^{\overline{A}}(\overline{z}), \quad \overline{A} = 1, \ldots, n, \end{aligned} \tag{2.1}$$

where f and $\overline{f}$ are arbitrary functions. We call a *W-surface* the corresponding manifold Σ. We shall see that its extrinsic geometry is directly related to W-transformations. It is convenient to introduce the matrix of inner products:

$$g_{i\overline{j}} \equiv \sum_A f^{(i)A}(z)\,\overline{f}^{(\overline{j})A}(\overline{z}), \quad 1 \leq i, j \leq n. \tag{2.2}$$

We use ∂ and $\overline{\partial}$ as shorthand for $\partial/\partial z$ and $\partial/\partial\overline{z}$, respectively. $f^{(i)A}$ stands for $(\partial)^i f^A$, and $\overline{f}^{(\overline{j})A}$ stands for $(\overline{\partial})^{\overline{j}}\overline{f}^A$. Later on we shall exhibit a particular parametrization of C^n, called W-parametrization, where the derivatives with i

or $\bar{j}$ larger than one will be replaced by first order derivatives in other variables, so that the covariance properties of the present discussion will become more transparent. This section is concerned with generic regular points of Σ where the Taylor expansions of f^A and $\overline{f}^{\bar{A}}$ give linearly independent vectors. Then $f^{(a)}$ and $\overline{f}^{(a)}$, $a = 1, \dots, n$, span the following

Definition 1. *Moving frame in* C^n. Consider the vectors $\boldsymbol{e}_a$, and $\bar{\boldsymbol{e}}_a$, $a = 1, \dots, n$, with components

$$e_a^A = \frac{1}{\sqrt{\Delta_a \bar{\Delta}_{a-1}}} \begin{vmatrix} g_{1\bar{1}} & \cdots & g_{a\bar{1}} \\ \vdots & & \vdots \\ g_{1\overline{a-1}} & \cdots & g_{a\overline{a-1}} \\ f^{(1)A} & \dots & f^{(a)A} \end{vmatrix}, \qquad e_a^{\bar{A}} = 0, \tag{2.3}$$

$$\bar{e}_a^A = 0, \qquad \bar{e}_a^{\bar{A}} = \frac{1}{\sqrt{\Delta_a \Delta_{a-1}}} \begin{vmatrix} g_{\bar{1}1} & \cdots & g_{\bar{a}1} \\ \vdots & & \vdots \\ g_{\bar{1}a-1} & \cdots & g_{\bar{a}a-1} \\ \overline{f}^{(1)\bar{A}} & \dots & \overline{f}^{(a)\bar{A}} \end{vmatrix}, \tag{2.4}$$

where Δ_a is the determinant

$$\Delta_a \equiv \begin{vmatrix} g_{1\bar{1}} & \cdots & g_{a\bar{1}} \\ \vdots & & \vdots \\ g_{1\bar{a}} & \cdots & g_{a\bar{a}} \end{vmatrix}. \tag{2.5}$$

One may verify that the moving frame defined above is orthonormal, that is,

$$(\boldsymbol{e}_a, \boldsymbol{e}_b) = (\bar{\boldsymbol{e}}_a, \bar{\boldsymbol{e}}_b) = 0, \quad (\boldsymbol{e}_a, \bar{\boldsymbol{e}}_b) = \delta_{a,b}. \tag{2.6}$$

The vectors $\boldsymbol{e}_1$ and $\bar{\boldsymbol{e}}_1$ are tangents to the surface, while the other vectors are clearly normals. Thus the Gauss–Codazzi equations may be derived by studying their derivatives along the W-surface Σ. One derives equations of the form

$$\begin{aligned} \partial \boldsymbol{e}_a &= \sum_b \omega_{za}^b \boldsymbol{e}_b, & \bar{\partial} \boldsymbol{e}_a &= \sum_b \omega_{\bar{z}a}^b \boldsymbol{e}_b, \\ \bar{\partial} \bar{\boldsymbol{e}}_a &= \sum_b \overline{\omega}_{\bar{z}a}^b \bar{\boldsymbol{e}}_b, & \partial \bar{\boldsymbol{e}}_a &= \sum_b \overline{\omega}_{za}^b \bar{\boldsymbol{e}}_b, \end{aligned} \tag{2.7}$$

which may be regarded as generalized Frenet–Serret formulae. Next we recall the

Definition 2. CP^n *target space*. The complex projective space C^n is defined to be the quotient of the space C^{n+1} by the equivalence relation

$$\boldsymbol{X} \sim \boldsymbol{Y}, \qquad \text{if} \quad X^A = Y^A \rho(Y), \quad \text{and} \quad \overline{X}^{\bar{A}} = \overline{Y}^{\bar{A}} \overline{\rho}(\overline{Y}), \tag{2.8}$$

where ρ and $\overline{\rho}$ are arbitrary functions of $n + 1$ variables.

Thus, the modification when going from C^n to CP^n is to use $n+1$ coordinates, so that now A and $\overline{A}$ run from 0 to n, and to write formulae that are covariant under rescaling. This is achieved by letting the indices of the matrix $g_{i\bar{j}}$ run from 0 to n in the CP^n formulae for the moving frame. For this one includes derivatives of order 0. Our basic result is the following:

Theorem 1. *Gauss–Codazzi equations. Define Toda fields by*

$$\phi_a = -\ln(\tau_a), \quad a = 1, \dots, n; \qquad \tau_a \equiv \begin{vmatrix} g_{00} & \cdots & g_{a0} \\ \vdots & & \vdots \\ g_{0a} & \cdots & g_{aa} \end{vmatrix}. \tag{2.9}$$

The integrability conditions of the Frenet–Serret equations for the embedding in CP^n *[analogous to eqs. (2.7)] coincide with the Toda equations associated with* A_n*:*

$$\partial\bar{\partial}\phi_i + \exp\left(\sum_{j=1}^{n} K_{ij}\phi_j\right) = 0. \tag{2.10}$$

K is the Cartan matrix associated with A_n. The functions τ_a relevant for CP^n are similar to the Δ_a of eq. (2.5) except that they include the derivatives of zeroth order.

In conclusion: The A_n Toda dynamics describes the extrinsic geometry of W-surfaces.

2.2. SOME BASIC FACTS ABOUT A_n TODA THEORIES

Its general solution is of the form

$$e^{-\phi_k} = \sum_{i_1<\cdots<i_k} \begin{vmatrix} \chi^{i_1} & \cdots & \chi^{i_k} \\ \chi^{(1)\,i_1} & \cdots & \chi^{(1)\,i_k} \\ \vdots & \cdots & \vdots \\ \chi^{(k-1)\,i_1} & \cdots & \chi^{(k-1)\,i_k} \end{vmatrix} \begin{vmatrix} \bar{\chi}^{i_1} & \cdots & \bar{\chi}^{i_k} \\ \bar{\chi}^{(1)\,i_1} & \cdots & \bar{\chi}^{(1)\,i_k} \\ \vdots & \cdots & \vdots \\ \bar{\chi}^{(k-1)\,i_1} & \cdots & \bar{\chi}^{(k-1)\,i_k} \end{vmatrix}, \tag{2.11}$$

where k runs from 1 to n. It is expressed in terms of n functions $\chi^1, \dots, \chi^n$ of z and n functions $\bar{\chi}^1, \dots, \bar{\chi}^n$ of $\bar{z}$. Upper indices between parentheses denote derivatives. These n functions χ^k ($\bar{\chi}^k$) are restricted to be solution of the same differential equation $\chi^{(n+1)k} - \sum_{l=0}^{n-1} U_{n+1-l}\chi^{(l)k} = 0$ ($\bar{\chi}^{(n+1)k} - \sum_{l=0}^{n-1} \overline{U}_{n+1-l}\bar{\chi}^{(l)k} = 0$). The set of potentials $\{U_l, l = 2, \dots, n+1\}$ and $\{\overline{U}_l, l = 2, \dots, n+1\}$, each generate a realization of the A_n W-algebra by Poisson brackets. These are non-linear generalizations of the Virasoro algebra (conformal transformations). The Toda dynamics is non-chiral, and this is why the W-algebra appears twice (for the holomorphic and anti-holomorphic components).

It follows that from the above geometrical derivation of the Toda equations, we may discuss the geometrical meaning of the W-transformations.

2.3. CONNECTION WITH THE WZNW MODELS

It is known, in general, that there is one W-algebra associated with each simple[#1] Lie algebra $\mathcal{G}$. This appears in several ways. First, as we have already seen, there is a Toda theory, and, hence, two PB realizations of the associated W-algebra for any given $\mathcal{G}$. On the other hand, consider the affine Lie algebra $\widetilde{\mathcal{G}}$ associated with $\mathcal{G}$. The associated non-chiral theory is the WZNW model whose quantum solutions are given by representations of $\widetilde{\mathcal{G}}$. It is possible to derive the Toda theory from the WZNW model by conformal reduction [8]. Here we have the

Theorem 2. *Gauss decomposition from moving frame. The moving-frame equations may be written as*

$$\boldsymbol{e}_a = \sum_{b\leq a} C_{ab}(z,\bar{z}) \sqrt{\frac{\tau_a}{\tau_{a+1}}}\, \boldsymbol{f}^{(b)}(z), \quad \text{with } C_{aa} = 1, \tag{2.12}$$

$$\bar{\boldsymbol{e}}_a = \sum_{b\leq a} A_{ba}(z,\bar{z}) \sqrt{\frac{\tau_a}{\tau_{a+1}}}\, \bar{\boldsymbol{f}}^{(b)}(\bar{z}), \quad \text{with } A_{aa} = 1. \tag{2.13}$$

The matrix $\theta = g^{-1}$ is such that

(i) It has the Gauss decomposition

$$\theta = ABC, \tag{2.14}$$

where A and C, which appear in eqs. (2.12) and (2.13) are triangular with diagonal elements equal to one, and

$$B = \exp\left(\sum_i h_i \phi_{i+1}\right),$$

where h_i are the Cartan generators.

(ii) It is a solution of the conformally reduced WZNW model associated with A_n.

3. Geometry of Toda hierarchy

The Toda equations are a subsystem of the Toda hierarchy [6]. (This is the non-chiral version of the fact that the Virasoro algebra is identical with the

[#1] The non-simple Lie algebras are trivially reduced to the simple case by separating the invariant subalgebras.

second Poisson bracket of KdV, and that W-algebras are obtained from KP hierarchies and Gelfand–Dicki brackets). Introduce the additional variables as coordinates in our geometrical embedding problem. This is best done using the free-fermion formalism. Let

$$\begin{aligned}[\psi_n, \psi_m]_+ &= [\psi_n^+, \psi_m^+]_+ = 0, \\ [\psi_n, \psi_m^+]_+ &= \delta_{n,m} \qquad (n, m = 0, 1, \ldots),\end{aligned} \tag{3.1}$$

$$\psi_n |\emptyset\rangle = 0, \qquad \langle\emptyset| \psi_n^+ = 0, \qquad \forall n. \tag{3.2}$$

We use the semi-infinite indices $n = 0, 1, 2, \ldots, \infty$ for the fermion operators. The vacuum states $|\emptyset\rangle$ and $\langle\emptyset|$ correspond to the no-particle states. The n-particle ground state is created from them in the standard way:

$$|n\rangle = \psi_{n-1}^+ \psi_{n-2}^+ \cdots \psi_0^+ |\emptyset\rangle, \quad \langle n| = \langle\emptyset| \psi_0 \psi_1 \cdots \psi_{n-1}. \tag{3.3}$$

The current operators,

$$J_n = \sum_{s=0}^{\infty} \psi_{n+s}^+ \psi_s, \qquad \overline{J}_n = \sum_{s=0}^{\infty} \psi_s^+ \psi_{n+s}, \tag{3.4}$$

will be taken as Hamiltonians as one does for the KP hierarchy. The role of these fermions may be understood as follows. Take the case where z is a complex variable. Then the embedding functions f^A are analytic, and each of them is entirely determined by its Taylor expansion around a single point of its analyticity domain. Its behaviour at any other point of its Riemann surface is fixed by analytic continuation. The following free-fermion formalism realizes this continuation automatically. Consider the Taylor expansions at the point z:

$$f^A(z + x) = \sum_{s=0}^{\infty} f^{(s)A}(z) \frac{x^s}{s!}, \quad \overline{f}^A(\overline{z} + \overline{x}) = \sum_{s=0}^{\infty} \overline{f}^{(s)A}(\overline{z}) \frac{\overline{x}^s}{s!}. \tag{3.5}$$

To these expansions we associate the free-fermion operators

$$\psi_{f^A(z)} = \sum_{s=0}^{\infty} f^{(s)A}(z)\, \psi_s, \quad \psi^+_{\overline{f}^A(\overline{z})} = \sum_{s=0}^{\infty} \overline{f}^{(s)A}(\overline{z})\, \psi_s^+. \tag{3.6}$$

The basic property of these operators is

Proposition 1. *Fermionic representation of chiral functions.*

(i) Any change of the Taylor-expansion point z, $\overline{z}$ can be absorbed by the action of the Hamiltonians J_1 and $\overline{J}_1$. In particular, one has

$$\psi_{f^A(z)} = e^{-J_1 z} \psi_{f^A(0)} e^{J_1 z}, \qquad \psi^+_{\overline{f}^A(\overline{z})} = e^{\overline{J}_1 \overline{z}} \psi^+_{\overline{f}^A(0)} e^{-\overline{J}_1 \overline{z}}. \tag{3.7}$$

(ii) The embedding functions are represented by the fermion expectation values

$$f^A(z) = \langle\emptyset| \psi_{f^A(z_0)} e^{J_1(z-z_0)} |1\rangle, \qquad \overline{f}^A(\overline{z}) = \langle 1| e^{\overline{J}_1(\overline{z}-\overline{z}_0)} \psi^+_{\overline{f}^A(\overline{z}_0)} |\emptyset\rangle. \tag{3.8}$$

Definition 3. *KP-parametrization of* CP^n. Given a chiral surface embedded into CP^n, the associated KP-parameters of the target space are $n+1$ variables $z^{(0)}$, $z^{(1)}=z, z^{(2)},\dots, z^{(n)}$, noted $[z]$, and $n+1$ variables $\bar z^{(0)}, \bar z^{(1)}=\bar z, \bar z^{(2)},\dots, \bar z^{(n)}$, noted $[\bar z]$. The change of coordinates from X^A, $\bar X^{\bar A}$ to $[z]$, $[\bar z]$ is defined by

$$X^A = f^A([z]), \qquad \bar X^{\bar A} = \bar f^{\bar A}([\bar z]), \tag{3.9}$$

where $f^A([z])$, and $\bar f^{\bar A}([\bar z])$, are the solutions of the equations

$$\frac{\partial}{\partial z^{(l)}} f^A([z]) = \frac{\partial^l}{\partial z^l} f^A([z]), \quad \frac{\bar\partial}{\partial \bar z^{(l)}} \bar f^{\bar A}([\bar z]) = \frac{\bar\partial^l}{\partial \bar z^l} \bar f^{\bar A}([\bar z]), \tag{3.10}$$

with the initial conditions $f^A([z]) = f^A(z)$ for $z^{(0)}, z^{(2)},\dots, z^{(n)}=0$, and $\bar f^{\bar A}([\bar z]) = \bar f^{\bar A}(\bar z)$ for $\bar z^{(0)}, \bar z^{(2)},\dots,\bar z^{(n)}=0$.

These coordinates coincide with the higher variables of the KP hierarchy. Indeed, their definition is most natural in the free-fermion language, where it is easy to see that

$$\begin{aligned} f^A([z]) &= \langle\emptyset|\psi_{f^A} \exp\Big(\sum_0^n J_s z^{(s)}\Big)|1\rangle, \\ \bar f^{\bar A}([\bar z]) &= \langle 1|\exp\Big(\sum_0^n \bar J_t \bar z^{(t)}\Big)\psi^+_{\bar f^{\bar A}}|\emptyset\rangle. \end{aligned} \tag{3.11}$$

The dependence on $[z]$ and $[\bar z]$ is dictated by the action of the higher currents J, $\bar J$, defined by eq. (3.4), that is, $J_1 z \to \sum_{i=0}^n J_i z^{(i)}$, $\bar J_1 \bar z \to \sum_{i=0}^n \bar J_i \bar z^{(i)}$ in eq. (3.8). The basic virtue of the KP coordinates is that they allow us to extend the previous discussion away from the W-surface (they parametrize at least a neighborhood of it) in such a way that is becomes covariant. In particular, the metric g has the following extension:

$$g_{i\bar j}([z],[\bar z]) = \sum_A \partial_i f^A([z])\bar\partial_j \bar f^{\bar A}([z]), \quad \partial_i \equiv \frac{\partial}{\partial z^{(i)}}, \quad \bar\partial_i \equiv \frac{\bar\partial}{\partial \bar z^{(i)}}. \tag{3.12}$$

Now, only first order derivatives appear. This expression coincides with the *true* Riemannian metric with respect to the KP coordinates.

3.1. W-TRANSFORMATIONS

A general infinitesimal W-transformation is a change of embedding functions which takes the form

$$\delta_W f^A(z) = \sum_{j=0}^n w^j(z)\partial^{(j)} f^A(z), \quad \delta_W \bar f^{\bar A}(\bar z) = \sum_{j=0}^n \bar w^j(\bar z)\bar\partial^{(j)}\bar f^{\bar A}(\bar z). \tag{3.13}$$

It is not difficult to show there exists a unique extension such that the differential equation (3.10) is left invariant. It is of the form

$$\delta_W f^A([z]) = \sum_r W^r([z])\partial_r f^A([z]),$$
$$\delta_W \overline{f}^{\overline{A}}([\overline{z}]) = \sum_r \overline{W}^r([\overline{z}])\partial_r \overline{f}^{\overline{A}}([\overline{z}]), \tag{3.14}$$

where W^r and $\overline{W}^r$ are functionals of w^j and $\overline{w}^j$, respectively. Only first order derivatives appear. Thus the W-transformations become extended as reparametrizations,

$$\delta_W z^{(r)} = W^r([z]), \quad \delta_W \overline{z}^{(r)} = \overline{W}^r([\overline{z}]). \tag{3.15}$$

They become particular diffeomorphisms of CP^n. Thus they are extended as linear transformations.

3.2. DYNAMICAL EQUATIONS

The followings topics are discussed in ref. [2].

The above functions τ_a when extended become tau-functions in the sense of Miwa–Jimbo–Sato [6].

The KP coordinates are related with a generalized moving frame, whose integrability condition is equivalent to the the well-known Zakharov–Shabat condition of the A_n Toda hierarchy.

The extension of the associated WZNW model gives solutions of a $2n$ dimensional generalization of the WZNW equations where the currents are replaced by the Christoffel symbols of the KP coordinates.

4. Singular points, global structure

At this point, it is useful to change the viewpoint, and make use of Grassmannians. This part draws much inspiration from ref. [9]

4.1. ASSOCIATED MAPPINGS

Definition 4. *Associated mappings.* Consider the family of osculating hyperplanes with contact of order k denoted $\mathcal{O}_k$ $(k = 1, \ldots, n)$ to the original W-surface. With CP^n as the target space, this family defines an embedding into the Grassmannian $G_{n+1,k+1}$, which we call the kth associated mappings to the original W-surface.

This formulation looks different, but is equivalent to the construction of the moving frame and only uses the intrinsic geometrics of the induced metrics for

$k = 1, \ldots, n$. In practice, what this means is that, instead of forming moving-frame vectors $\boldsymbol{e}_k$ out of $\boldsymbol{f}, \ldots, \boldsymbol{f}^{(k)}$ ($k = 1, \ldots, n$), we consider the nested osculating planes $\mathcal{O}_1 \subset \mathcal{O}_2 \subset \cdots \subset \mathcal{O}_n$. It is obvious that those two have the same information. It is well known that the Grassmannians are Kähler manifolds. The induced metric on the kth associated surface in $G_{n+1,k+1}$ is simply

$$g^{(k)}_{z\bar{z}} = \partial\bar{\partial} \ln \tau_{k+1}(z, \bar{z}), \quad g^{(k)}_{zz} = g^{(k)}_{\bar{z}\bar{z}} = 0, \tag{4.1}$$

so that the Toda field $\phi_{k+1} \equiv -\ln(\tau_{k+1})$ appears naturally. Thus $-\phi_{k+1}$ is equal to the Kähler potential of the kth associated surface. At this point, it is very clear that by considering the associated surfaces, we can restrict ourselves to intrinsic geometries. In the discussion of section 2, the Toda equation came from the Gauss–Codazzi equation. Here, it is equivalent to the local Plücker formula

$$R^{(k)}_{z\bar{z}} \sqrt{g^{(k)}_{z\bar{z}}} = -g^{(k+1)}_{z\bar{z}} + 2g^{(k)}_{z\bar{z}} - g^{(k-1)}_{z\bar{z}}, \tag{4.2}$$

where $R^{(k)}_{z\bar{z}}$ is the only non-vanishing component of the intrinsic Riemann tensor on the kth surface.

4.2. THE INSTANTON NUMBERS OF A W-SURFACE

A key point in the coming discussion is the use of topological quantities that are instanton numbers. W-surfaces are instantons of the associated non-linear σ-model. The general situation is as follows. W-surfaces are characterized by their chiral parametrizations which thus satisfy the Cauchy–Riemann relations. These are self-duality equations so that the coordinates of a W-surface define fields that are solutions of the associated non-linear σ-model, with an action equal to the topological instanton number. For a general Kähler manifold M with coordinates ξ^μ and $\bar{\xi}^{\bar{\mu}}$, and metric $h_{\mu\bar{\mu}}$, the action associated with 2D manifolds of M with equations $\xi^\mu = \varphi^\mu(z, \bar{z})$, and $\bar{\xi}^{\bar{\mu}} = \bar{\varphi}^{\bar{\mu}}(z, \bar{z})$ is

$$S = \frac{1}{2} \int \mathrm{d}^2x \; h_{\mu\bar{\mu}} \, \partial_j \varphi^\mu \partial_j \bar{\varphi}^{\bar{\mu}}. \tag{4.3}$$

In this short digression we let $z = x_1 + \mathrm{i}x_2$, and $\partial_j = \partial/\partial x_j$. The instanton number is defined by

$$Q = \frac{\mathrm{i}}{2\pi} \int \mathrm{d}^2x \; \epsilon_{jk} \; h_{\mu\bar{\mu}} \, \partial_j \varphi^\mu \partial_k \bar{\varphi}^{\bar{\mu}}. \tag{4.4}$$

For W-surfaces and their associated surfaces, $\bar{\partial}\varphi^\mu = \partial\bar{\varphi}^{\bar{\mu}} = 0$, and one has $S = \pi Q$. Q is proportional to the integral of the determinant of the induced metric, that is,

$$Q = \frac{\mathrm{i}}{2\pi} \int \mathrm{d}^2x \; \partial\bar{\partial} \ln \tau_1 .$$

Moreover we may also apply formula eq. (4.4) to the kth associated surface. This gives

Definition 5. *Higher instanton numbers of the W-surface.* The kth instanton number of the W-surface Q_{k+1} is defined by

$$Q_{k+1} \equiv \frac{\mathrm{i}}{2\pi} \int_{\Sigma} \mathrm{d}z\, \mathrm{d}\bar{z}\, g^{(k)}_{z\bar{z}}, \quad k = 1, \ldots, n-1. \tag{4.5}$$

Its topological nature is obvious from eq. (4.1), which shows that the integrand is indeed a total derivative. The collection of the kth instanton numbers together with the original one $Q \equiv Q_1$ gives a set of topological quantities which characterize the global properties of the original W-surface.

4.3. GLOBAL CLASSIFICATION OF W-SURFACES

In the main section 2, we have constructed the moving frames at the point where the tau-functions are regular. When those functions become irregular, we meet an obstruction to deriving the moving frames. In the WZNW language, this signals that there appears a global obstruction to the Gauss decomposition. Toda equations should be modified at these points. In the following, we study the structure of such singularities.

4.4. GAUSS–BONNET THEOREM FOR W-SURFACES

For isolated singularities [#2], the obstruction to constructing the moving frame may be reduced to the vanishing of certain terms in the Taylor expansion. The latter is characterized by the ramification indices β_k which are integers. Apply the Gauss–Bonnet theorem for each of the kth associated surfaces by computing $\int_{\Sigma_\epsilon} R^{(k)}_{z\bar{z}} (g^{(k)}_{z\bar{z}})^{1/2}$. The integral is first computed over Σ_ϵ, where small neighborhoods of singularities are removed. The ramification indices at a singularity were defined so that at a singular point the induced metric of the kth associated surface behaves as

$$g^{(k)}_{z\bar{z}} \sim (z - z_0)^{\beta_k(z_0)} (\bar{z} - \bar{z}_0)^{\bar{\beta}_k(\bar{z}_0)} \tilde{g}^{(k)}_{z\bar{z}}, \tag{4.6}$$

where $\tilde{g}^{(k)}_{z\bar{z}}$ is regular at z_0, $\bar{z}_0$. Since we do not assume that $\overline{f(z)} = \bar{f}(\bar{z})$, $\beta_k(z_0)$ and $\bar{\beta}_k(\bar{z}_0)$ may be different. By letting $\epsilon \to 0$, one sees, that the contribution of the singularities to the Gauss–Bonnet formula is proportional to the kth ramification index

$$\beta_k \equiv \frac{1}{2} \sum_{(z_0, \bar{z}_0) \in \Sigma} \Big(\beta_k(z_0) + \bar{\beta}_k(\bar{z}_0) \Big). \tag{4.7}$$

[#2] If there is a cut with a finite number of sheet, one takes the appropriate covering.

The contribution of the regular part does not depend upon k, since changing k there is equivalent to using a different complex structure, while the result is equal to the Euler characteristic that does not depend upon it. The Gauss–Bonnet theorem for the kth associated surface finally gives

$$\frac{\mathrm{i}}{2\pi}\int_{\Sigma} \mathrm{d}z\,\mathrm{d}\bar{z}\,R^{(k)}_{z\bar{z}}\sqrt{g^{(k)}_{z\bar{z}}} = 2 - 2g + \beta_k. \tag{4.8}$$

Combining these last relations with eqs. (4.2), one arrives at

Theorem 3. *Global Plücker formulae. The genus g of a W-surface is related to its instanton numbers and ramification indices by the relations*

$$\begin{aligned} 2 - 2g + \beta_k &= 2Q_k - Q_{k+1} - Q_{k-1}, \quad k = 1,\ldots,n, \\ Q_{n+1} &\equiv 0, \qquad Q_0 \equiv 0. \end{aligned} \tag{4.9}$$

4.5. SIMPLE EXAMPLE

Consider the case of Liouville theory, for which one has $n = 1$. The simplest chiral surface corresponds to

$$f^0 = 1, \qquad f^1 = z; \qquad \bar{f}^0 = 1, \qquad \bar{f}^1 = \bar{z};$$
$$\tau_1 = 1 + z\bar{z}, \qquad \tau_2 = 1.$$

The instanton number is thus

$$Q = \frac{\mathrm{i}}{2\pi}\int \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{(1 + z\bar{z})^2} = 1.$$

On the other hand, one has

$$R_{z\bar{z}}\sqrt{g_{z\bar{z}}} = -\partial\bar{\partial}\ln(g_{z\bar{z}}) = \frac{2}{(1 + z\bar{z})^2}$$

so that

$$\frac{\mathrm{i}}{2\pi}\int \mathrm{d}z\,\mathrm{d}\bar{z}\,R_{z\bar{z}}\sqrt{g_{z\bar{z}}} = 2Q = 2,$$

and one verifies that the above formulae indeed hold with vanishing genus and ramification index. In this example, the Liouville solution coincides with the metric of the Lobachevski half-plane. Upon quantization, it corresponds to the SL(2, $\mathbb{C}$) invariant vacuum of the Liouville theory.

References

[1] J.-L. Gervais and Y. Matsuo, W geometries, Phys. Lett. B 274 (1991) 309.
[2] J.-L. Gervais and Y. Matsuo, A_n-W-geometry, preprint LPTENS-95/35, hep-th 9201026, Commun. Math. Phys., to be published.

[3] V.A. Fateev and A.B. Zamolodchikov, Conformal quantum field theory models in 2 dimensions having Z_3 symmetry, Nucl. Phys. B 280 [FS18] (1987) 644, and references therein.
[4] A. Bilal and J.-L. Gervais, Systematic approach to conformal systems with extended Virasoro symmetries, Phys. Lett. B 206 (1988) 412; Extended $C = \infty$ conformal systems from classical Toda field theories, Nucl. Phys. B 314 (1989) 646; Systematic construction of conformal theories with higher-spin Virasoro symmetries, Nucl. Phys. B 318 (1989) 579.
[5] J.-L. Gervais, Infinite family of polynomial functions of the Virasoro generators with vanishing Poisson brackets, Phys. Lett. B 160 (1985) 277;
K. Yamagishi, The KP hierarchy and extended Toda theories, Phys. Lett B 205 (1988) 406;
I. Bakas, The Hamiltonian structure of the spin-4 operator algebra, Phys. Lett. B 213 (1988) 313.
[6] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, Transformation groups for soliton equations, in: *Proc. RIMS Symp. on Non-linear Integrable Systems—Classical Theory and Quantum Theory* (Kyoto, Japan, 1981) (World Scientific, Singapore, 1983), and references therein.
[7] V.G. Drinfeld and V.V. Sokolov, Lie algebras and equations of the Korteweg–de-Vries type, J. Sov. Math. 30 (1985) 1975.
[8] P. Forgács, A. Wipf, J. Balog, L. Fehér and L. O'Raifeartaigh, Liouville and Toda theories as conformally reduced WZNW theories, Phys. Lett. B 227 (1989) 214;
J. Balog, L. Fehér, L. O'Raifeartaigh, P. Forgács and A. Wipf, Toda theory and W-algebra from gauged WZNW point of view, Ann. Phys. (NY) 203 (1990) 76; Kac–Moody realizations of W algebras, Phys. Lett. B 244 (1990) 435;
L. O'Raifeartaigh, P. Ruelle, I. Tsutsui and A. Wipf, W-algebras for generalized Toda theories, Dublin preprint DIAS-STP-91-03, Commun. Math. Phys., to appear;
L. Fehér, L. O'Raifeartaigh, P. Ruelle, I. Tsutsui and A. Wipf, On the general structure of Hamiltonian reductions of the WZNW theory, preprint DIAS-STP-91-29, and references therein.
[9] P. Griffiths and S. Harris, *Principles of Algebraic Geometry* (Wiley-Interscience, New York, 1978).